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Abstract

We study a game in which players are randomly matched to play a prisoner's dilemma game in each period. The previous works consider whether community enforcement works and cooperation can be sustained in random matching games as repeated games by fixed players and show some positive results. These results are given on the premise that a player perfectly observes the opponent's action in her pair. However, the plausibility of perfect observability between paired players is open to consideration. In contrast to the previous works, we consider an environment where the premise does not hold. We focus on the case where each player cannot observe the opponent's action in her pair but can observe an imperfect and public signal about it. In addition, we set up an environment where local information processing is available and players can make use of social norms. Under the environment, noisy information is exchanged among players and decisions are made on noisy information. Through examinations of numerical examples, even in such an environment we find "norm equilibria", which Okuno-Fujiwara and Postlewaite (1995) propose. Norm equilibria can sustain partial cooperation.

Keywords: Random Matching Games, Prisoner's Dilemma, Imperfect Public Monitoring, Social Norms, Norm Equilibrium.

JEL Classification Number: C72,C73.

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1 Introduction

We study a game in which players are randomly matched to play a prisoner's dilemma game in each period. In random matching games, a player is unlikely to interact with the same player. If a player deviates, the opponent player has a limited ability to punish the deviator. Thus, it may be difficult to sustain cooperation through a personal enforcement. Even in such environments, cooperation can be sustained if community enforcement, in which a deviation causes punishments by other players, works. In random matching games, a player only observes the action played by her opponent but cannot observe the actions played in other pairs in each period. Information available to players in order to coordinate their actions is very limited. Thus, some additional artifice is necessary for community enforcement to work. Several papers have shown some positive results on this issue.

The ways having been considered to overcome the difficulty are categorized into the following three groups. In the first group, it is considered to make use of the contagious strategy, which is considered as an extension of the trigger strategy. Kandori (1992) shows that contagious equilibria can sustain cooperation in the extreme case where players only observe the actions in the stage games in which they have been involved and do not use additional information. Ellison (1994) also assumes that players follow a contagious strategy and shows that if all players can observe a public random variable, which helps players' actions to coordinate, cooperation can be sustained under milder condition than Kandori (1992) assumes. However, the assumption that players use no additional information about their opponents may be demanding. In the second and third groups, the assumption is relaxed in some ways.

In the second group, researchers assume an environment where local information processing is available in a society. Okuno-Fujiwara and Postlewaite (1995) put forward a proposal of a type of information transmission as follows. Each player has a status which gives the opponent partial information about the actions played by her until the previous period. Before playing the stage game, each player observes the opponent's status. The statuses of paired players and the actions chosen by both the players in the period determine their statuses in the next period. If local information processing is available, social norms are useful to facilitate coordination among players. They show that cooperation can be sustained in the society with a continuum of population if players follow a social norm which prescribes cooperative behavior and sanction rules for the statuses of paired players. Kandori (1992) extends the environment where local information processing is available to a society with finite population and shows that cooperation can be sustained by means of a social norm which is different from what Okuno-Fujiwara and Postlewaite (1995) propose. In the third group, it is assumed that each player is able to know all or a part of actions which her opponent chose in the past. Takahashi (2010) and Deb (2020) apply the "belief-free" equilibria to such environments and show that cooperation can be sustained.

In random matching games, each player gets less information about the actions by players outside her pair than in her pair. The feature is essential and unquestionable in the literature. On the other hand, little attention has been attracted to the plausibility of the premise that each player perfectly observes her opponent's action in her pair.[1] The previous works are based on the premise in common. Information arises from actions which players actually observe. As a result, decisions are made on correct information. However, in reality, everyone makes mistakes. In our context, players can make wrong choices or misjudge the opponents' actions. The plausibility of perfect observability between paired players is open to consideration. In contrast to the previous works, we consider an environment where the premise does not hold. We focus on the case where each player cannot observe the opponent's action in her pair but can observe an imperfect and public signal about it. More concretely, we assume that the stage game played by paired players is a prisoner's dilemma game whose monitoring structure has two public signals.[2] In addition, following the approach of the second group, we set up an environment where local information processing is available and players can make use of social norms. Through examinations of numerical examples, we find "norm equilibria" which can sustain partial cooperation.

Now we present an outline of our model. We consider a large society with a continuum of population. All players in the society play a variant of random matching game continued infinitely, where time is discrete. All players are randomly matched at the beginning of each period. Then, paired players play a partnership game, which is ex-ante identical to a prisoner's dilemma game. In the partnership game, paired players cannot observe their opponents' actions each other. Instead, they observe a public signal which depends on actions chosen by them in the period. At the end of each period, all pairs dissolve.

[1] In the field relevant to random matching games, Greif (2006) refers to imperfect monitoring in the study about the Maghribi traders' coalition. Maurer and Sharma (2001) build a model with imperfect monitoring in order to explain lender's behavior under insecure property rights in Mexico's early industrialization. However, their models are composed of one-sided prisoner's dilemma games. Introducing imperfect monitoring has a limited implication such that cooperation phases and punishment phases are mixed on the equilibrium path.

[2] Following Mailath and Samuelson (2006), we use the terminology.

Our study relies heavily on the concept of norm equilibrium which Okuno-Fujiwara and Postlewaite (1995) propose. In general, the term "social norms" means standards of behavior that are typical or accepted within a society. In this literature, the term is often used to designate the combination of cooperative behavior and sanction rules. Under the concept of norm equilibrium, we redefine them as social standards of behavior, which are related to the statuses of paired players (henceforth, the pair's status). Thus, the social standards of behavior are identical to what we call Markov strategies. Okuno-Fujiwara and Postlewaite (1995) require that a social norm includes a transition rule of a player's status in addition to a social standard of behavior. In other words, a social norm is a pair consisting of a social standard of behavior and a transition rule of a player's status. Based on the definition of social norms, they give a definition of the norm equilibrium. The requirements that a social norm is called a norm equilibrium are (i) to follow the social standard of behavior is in the interest of each player in the society and (ii) the distribution of status levels (henceforth, the status distribution) in the society is stationary. Thus, players need two types of information. One is local information about the pair's status. The other is about the status distribution. The requirement (i) is the relaxation of ordinary equilibrium concepts in which players are assumed to pursue their own interests rationally. The requirement (ii) means that the society is in the steady state in an equilibrium.

The reason that the requirement (ii) is needed is peculiar to our model. The assumption of imperfect public monitoring implies that each player's status cannot be related to the actions observed in her pair. Instead, we assume that each player's status is related to a signal happened in her pair. Thus, noisy information is exchanged among players and decisions are made on noisy information. In addition, a player's status level changes according to a happened signal and players with different status levels are blended in the society. The status distribution, in principle, changes over time and influences each player's payoff. If the status distribution changes, then players may change their behavior. Thus, we need to assume that each player has a knowledge of the status distribution. Under the settings, we include the stationarity of players' behavior and the status distribution in the equilibrium concept.

We pick up some numerical examples of social norms which satisfy the conditions of norm equilibrium, i.e., the requirements (i) and (ii). Through examinations of these examples, we get the following results about norm equilibria in our model. First, multiple equilibria exist and various payoff levels can be realized in equilibria. Second, norm equilibria can sustain partial cooperation and punishments happen on the equilibrium path. Third, the maximum payoff sustained in norm equilibria can be greater than under the repeated games whose monitoring structure has two public signals. This result implies that there exist equilibria in which relatively high payoffs are realized even in an environment where players only exchange noisy information. Fourth, all players keep following the same pattern of behavior. The status distribution is unchanged in the steady state. In other words, as long as all players follow the actions prescribed by a social norm, the status distribution stays in the steady state. Such a situation is maintained in an equilibrium.

We make two remarks about our norm equilibria. The first remark is about off the equilibrium path. There exist a continuum of players and players are randomly matched in our model. Even if a negligible set of players deviate, the status distribution remains in the steady state. As a result, the action prescribed by the social norm is kept being optimal for each player. The second remark is about an implication of the norm equilibrium. Okuno-Fujiwara and Postlewaite (1995) stress that social norms help players coordinate their actions under the condition that information is limited. Given a particular situation, a player predicts her opponent's behavior on the basis of the prevailing social norm and chooses an optimal action depending on the prediction. However, they assume that observability among paired players is perfect. As a result, full cooperation is sustained in equilibria and all players in the society have one status, face one situation, and choose one action on the equilibrium path. Behavior featuring social norms is described off the equilibrium path rather than on the equilibrium path. As a result, the signification of requirement (ii) is blurred. In our imperfect monitoring settings, players with multiple statuses are blended in an equilibrium. They face multiple situations and behave according to each situation on the equilibrium path. We appropriately describe behavior featuring social norms on the equilibrium path and make the signification of requirement (ii) clear.

Recently, Heller and Mohlin (2018) consider an environment on the same assumption as the third group, in which a player can privately acquire information about a part of past actions chosen by the opponent. They incorporate some perturbations into the above environment and show that cooperation can be sustained even in an environment where there exist a fraction of irrational players in the society. Their modification of a random matching model is different from ours but their study is closely related to ours in some respects. The first point is about requirements imposed in the equilibrium concept. They include the stationarity of players' behavior and the status distribution in the equilibrium concept.[3] The

[3]Under the settings relevant to random matching games, Ghosh and Ray (1996) and Fujiwara-Greve and Okuno-Fujiwara (2009) impose the stationarity of the society as a requirement of their equilibria.

second point is about a monitoring structure assumed in the model. In random matching games, interactions between players are infrequent. Thus, distinguishing whether each interaction succeeds or fails is easier and more practical than identifying who deviates. They take notice of this point and propose a monitoring structure named "observing conflicts". The monitoring structure means that information which each player acquires about past actions chosen by the opponent is whether mutual cooperation was or not. The case of two public signals is considered as a generalization of observing conflicts. The third point is about player's knowledge of the status distribution in the society. Their equilibria are based on the assumption that players know the distribution of strategies prevailing in the society. As Okuno-Fujiwara and Postlewaite (1995) also stress, the requirement about knowledge of players is much weaker than the assumption that all the details of every interaction are common knowledge. However, there is room for some discussions for the assumption. We will mention this point in the Concluding Remarks.

The paper is organized as follows. In Section 2, we present a description of the model and an example of social norms. Using the example, we explain the concept of norm equilibrium. In Section 3, we consider the characteristics of norm equilibria in our model by investigating three examples of social norms. We examine the payoff attained in the norm equilibrium and the condition that social norms are norm equilibria. In Section 4, we discuss the set of social norms sustained in equilibria and the set of attainable payoffs. In Section 5, we conclude the paper and mention some remaining issues.

2 Model

We present the following model. A society consists of a set of players I, which is identical with a continuum between 0 and 1, i.e., $I \equiv [0, 1]$. Each player plays a random matching game Γ, which we give a detailed description in the following subsections. Game Γ is continued infinitely, where time is discrete and denoted as $t = 1, 2, 3, \cdots$.

2.1 Stage Game

The stage game of game Γ is constructed as follows. All players are uniformly randomly matched at the beginning of the period. Then, paired players play a partnership game. This partnership game is based on Fudenberg, Levine, and Maskin (1994) in order to compare the payoffs attained in the equilibria of our model with theirs. At the end of the period, all current pairs dissolve.

In the partnership game, the set of actions for player $i \in I$ is $A_i \equiv \{E, S\}$, the actions E and S stand for "effort" and "shirk", respectively. A particular feasible action for i is expressed by $a_i \in A_i$. The notation j expresses an opponent to i in a pair. We also use the notation $a \in A \equiv A_i \times A_j$, where $a \equiv (a_i, a_j)$.

At the end of the partnership game, paired players observe a public signal $y \in \{\bar{y}, \underline{y}\} \equiv Y$. The signal $\bar{y}$ means the "good" signal, while the signal $\underline{y}$ means the "bad" signal. The probability that the signal y is realized given the action profile a is denoted by $\rho(y|a)$. In this paper, we consistently assume that the values of the function $\rho(y|a)$ are concretely determined as follows:

$$(\rho(\bar{y}|a), \rho(\underline{y}|a)) = \begin{cases} (2/3, 1/3), & if \quad a = (E, E), \\ (1/3, 2/3), & if \quad a = (S, E) \text{ or } (E, S), \\ (0, 1), & if \quad a = (S, S). \end{cases}$$

This probability distribution means that the bad signal $\underline{y}$ happens with a positive probability even if both players exert effort. Additionally, the more players exert effort, the higher is the probability that the good signal is realized.

A revenue for each pair is a function of y, which is denoted by $R(y)$. We assume that $(R(\bar{y}), R(\underline{y})) = (12, 0)$ and paired players divide the realized revenue equally. Player i's cost $c_i(a_i)$ depends on whether he makes effort or shirk. We assume $(c_i(E), c_i(S)) = (3, 0)$. Then, player i's expected payoff $\pi_i(a)$ is calculated as follows:

$$\pi_i(a) = \rho(\bar{y}|a)\frac{R(\bar{y})}{2} + \rho(\underline{y}|a)\frac{R(\underline{y})}{2} - c_i(a_i),$$

and the expected payoff matrix is the following prisoner's dilemma game.

	E	S
E	1, 1,	−1, 2
S	2, −1	0,0

Figure 1. The ex-ante payoffs for the partnership game.

2.2 Payoffs of Game Γ

Each player discounts his future payoffs with a common discount factor $\delta \in [0, 1)$. Because our concern is about a player's decision in the steady state, continuation payoffs are mainly used in our analysis. So we only give a definition of the continuation payoff. A player's continuation payoff of the game Γ is the normalized sum of discounted payoffs from the stage games which she plays from the period on. Concretely, the continuation payoff in period t from the infinite sequence of payoffs $(v_t, v_{t+1}, v_{t+2}, \cdots)$ is given by

$$v = (1-\delta)\sum_{s=t}^{\infty} \delta^{s-t} v_s.$$

Unless any confusion arises, we simply write a payoff instead of a continuation payoff hereafter.

2.3 Description of Social Norms

Following Okuno-Fujiwara and Postlewaite (1995), we define a social norm as a pair consisting of a social standard of behavior and a transition rule of a player's status. Through this paper, we consistently focus on the case where all players in the society follow the same social norm in an equilibrium.

To describe a player's status formally, we incorporate state variables into our model. A state variable for player i is expressed by θ_i, whose value is g or b. Formally, $\theta_i \in \{g, b\} \equiv \Theta_i$. A status of a pair (i, j) is expressed by θ. More formally, $\theta = (\theta_i, \theta_j) \in \Theta_i \times \Theta_j \equiv \Theta$. These state variables are observable between paired players. In other words, the state variables correspond to statuses as information transmission contrivances. A player's status is attached to each player and give the opponent a part of information about the actions chosen by the player until the previous period. Additionally, we assume that the status cannot be manipulated by the owner. This assumption means that information is transmitted honestly.

A social standard of behavior and a transition rule of a player's status are respectively described as an output function and a transition function. We express player i's choice as the output function $f_i : \Theta \to A_i$.[4] Thus, the action which a social standard of behavior prescribes to a player depends on the current pair's status. The transition function identifies the next status of player i, given the current pair's status and the realized public signal. We

[4]In this paper, we assume that all players only choose pure actions. Thus, we use a notation in this way.

express it as $\tau_i : \Theta \times Y \to [0,1]$, where the value of $\tau_i(\theta, y)$ is the probability of transition to a status g, given θ and y. Thus, both a social standard of behavior and a transition rule depend only on noisy information.

Now we present a combination of an output function and a transition function, i.e., an example of social norms as follows.

Norm 1.

$$f_i(\theta) = \begin{cases} E, & if \quad \theta = (g,g),(b,g), \\ S, & if \quad \theta = (g,b),(b,b). \end{cases}$$

$$\tau_i(\theta, y) = \begin{cases} 1, & if \qquad (\theta, y) = ((g,g),\bar{y}),((g,b),\bar{y}),((g,b),\underline{y}),((b,g),\bar{y}), \\ 0, & if \qquad (\theta, y) = ((g,g),\underline{y}),((b,g),\underline{y}),((b,b),\bar{y}),((b,b),\underline{y}). \end{cases}$$

In process of setting up this example, we intend to make it as simple as possible in the light of the nature of social norms. What Norm 1 aims at are the followings. Regardless of the actions which the opponent has chosen until the previous period, each player chooses the action depending only on the current pair's status. Each player exerts effort when the status of the opponent is g or shirks when it is b. A distinctive feature of the output function is that g status player (henceforth, g player) gets a payoff of 2 while b status player (henceforth, b player) gets -1 when the pair's status is (g,b) or (b,g). In other words, asymmetric punishments, which gives a reward to g player while punishing b player, are implemented. Partial cooperation can be sustained by asymmetric punishments. In principle, the value of transition function $\tau_i(\theta, y)$ is assumed to be higher if player i follows the action prescribed by the output function than otherwise. When the pair's status is (g,b) or (b,g), g player's status is exceptionally unchanged regardless of a happened signal.

2.4 Steady States and Definition of Norm Equilibrium

In game Γ, social norms prevailing in the society influence a transition of each player's status. As a result, the status distribution in the society depends on social norms prevailing in the society. We explain this point by using Norm 1. Let $\mu_t \in [0,1]$ be the ratio of the population of g players to all players in period t, i.e., a variable describing the status distribution in period t. We assume that μ_t is known to all players. This assumption means that each player can obtain information about how many players in the society follow the prescribed social norm.

Suppose that the status distribution is μ_t and all players follow Norm 1. The ratio of pairs whose statuses are (g, g) is μ_t^2 in period t. Then, the signal $\bar{y}$ is realized in 2/3 of these pairs and their statuses remain g in the next period. On the other hand, the signal $\underline{y}$ is realized in 1/3 of these pairs and their statuses vary from g to b. By considering the other cases similarly, we find that the transition of μ_t follows the next equation.

$$\mu_{t+1} = \frac{2}{3}\mu_t^2 + \mu_t(1-\mu_t) + \frac{1}{3}\mu_t(1-\mu_t). \tag{1}$$

In Figure 2, the graph of equation (1) is drawn.

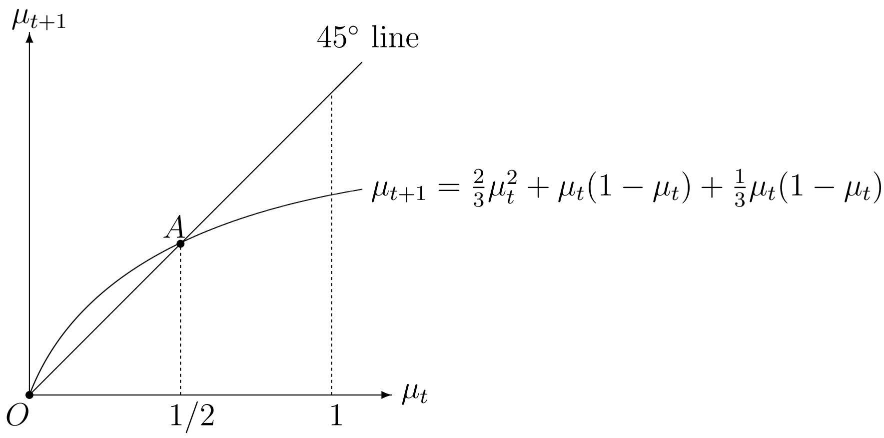

Figure 2. The transition of μ_t and the steady states.

In equation (1), there exist two steady states in which μ_t equals μ_{t+1}. One corresponds to the point O and the other corresponds to the point A in Figure 2. The point A is stable while the point O is unstable. All players' statuses are b and all players choose S at the point O. Thus, cooperative behavior never happens. In the case where players follow Norm 1, such a noncooperative outcome is of less interest to us. So we focus our analysis on the stable steady state, i.e., the point A. We denote the status distribution in the steady state by μ^* hereafter. Substituting the condition $\mu_t = \mu_{t+1}$ into equation (1), we get $\mu^* = 1/2$.

It is clear from equation (1) that the transition of μ_t depends on the current status distribution and actions chosen by all players in the society. Thus, the payoff of game Γ depends not only on social norms which prescribes players' actions but also on the current status distribution. With this in mind, the equilibrium in game Γ is defined as follows.

Definition 1 *A social norm* (f_i^*, τ_i^*) *and a status distribution* μ^* *are called a norm equilibrium of game* Γ *if* *(i) given τ_i^*, f_i^* is an optimal action for any i, θ, and f_i at μ^*,*

(ii) (f_i^*, τ_i^*) is stationary at μ^*.

This is none other than the definition of the norm equilibrium which Okuno-Fujiwara and Postlewaite (1995) propose. We refer to two remarks on the equilibrium concept. First, as stated in Introduction, the status distribution changes and influences each player's payoff in our model. Thus, each player's decision is consistent with the status distribution in an equilibrium. So we need the requirement (ii). Second, following a social norm is still optimal if negligible players' deviations happen. The assumptions of a continuum of population and random matching guarantee that each player's belief about μ_t is unchanged by such deviations. Thus, we are relieved from checking players' incentives off the equilibrium path.

3 Analyses Based on Numerical Examples

Using some numerical examples, we analyze norm equilibria in game Γ that we set up in the previous section. First, we consider Norm 1 which is presented as an example in subsection 2.3. The payoff which is realized under Norm 1 is equal to the maximum attainable payoff in the repeated game which we adopt as a benchmark. The benchmark is given in the next subsection. Then, we present Norms 2 and 3 as other social norms. Norms 2 and 3 are modifications of Norm 1 and they can achieve more efficient payoffs than Norm 1.

3.1 A Benchmark

Before considering Norms 1, 2, and 3, we set a benchmark to compare the payoffs attainable in equilibria. We adopt the numerical example, which Fudenberg, Levine, and Maskin (1994) give consideration to, as a benchmark. In the repeated game, the component game is the same as being set up in subsection 2.1. The repeated game is well known as the typical case that efficiency cannot be attained under imperfect public monitoring. The following fact is given in this repeated game.

Fact 1 *For any $\delta \in [0, 1)$, the sum of the average discounted payoffs of both players is no more than 1 in any equilibrium.*

See Fudenberg, Levine, and Maskin (1994) for a proof. Fact 1 means that the maximum attainable payoff profile is much lower than the efficient level including the profile $(1, 1)$ and a player's payoff is no more than $1/2$ on average.

3.2 Norm 1

We have already shown that the steady state μ^* equals $1/2$ if all players in the society follow Norm 1. In this subsection, we calculate the payoff which a player gets in the steady state. Then, we derive and check incentive compatible conditions in each pair's status.

We denote the payoff which a player gets in a pair's status θ by v_θ. Let us consider the case $\theta = (g, g)$ as a clue to calculate the equilibrium payoff. The payoff v_{gg} is related to ones in other pairs' statuses as follows:

$$v_{gg} = (1-\delta) + \delta\{\frac{2}{3}(\frac{1}{2}v_{gg} + \frac{1}{2}v_{gb}) + \frac{1}{3}(\frac{1}{2}v_{bg} + \frac{1}{2}v_{bb})\}. \tag{2}$$

When both players follow Norm 1, they choose E and get the expected payoff 1 in this period. At the end of the period, the signal $\bar{y}$ occurs with probability $2/3$ and their statuses remain g in the next period, while the signal $\underline{y}$ occurs with probability $1/3$ and their statuses turn to b in the next period. In addition, since $\mu^* = 1/2$, regardless of player's own status, each player is paired with g player with probability $1/2$ and with b player with probability $1/2$ in the next period. Thus, the above equation holds. Similarly, the payoffs in other pairs' statuses are described as follows:

$$v_{gb} = 2(1-\delta) + \delta(\frac{1}{2}v_{gg} + \frac{1}{2}v_{gb}), \tag{3}$$

$$v_{bg} = -(1-\delta) + \delta\{\frac{1}{3}(\frac{1}{2}v_{gg} + \frac{1}{2}v_{gb}) + \frac{2}{3}(\frac{1}{2}v_{bg} + \frac{1}{2}v_{bb})\}, \tag{4}$$

$$v_{bb} = \delta(\frac{1}{2}v_{bg} + \frac{1}{2}v_{bb}). \tag{5}$$

Solving the simultaneous linear equations (2)-(5), we obtain the payoffs in all types of pair's status as follows.

$$\begin{aligned}&(v_{gg}, v_{gb}, v_{bg}, v_{bb})\\ &= (\frac{6-5\delta}{6-4\delta}, \frac{12-11\delta}{6-4\delta}, \frac{(-2+\delta)(-3+4\delta)}{-(6-4\delta)}, \frac{(3-4\delta)\delta}{-(6-4\delta)}).\end{aligned} \tag{6}$$

Using the payoffs in all types of pair's status, we define a kind of payoff of game Γ as follows:

$$\bar{v} = (\mu^*)^2 v_{gg} + \mu^*(1-\mu^*)v_{gb} + (1-\mu^*)\mu^* v_{bg} + (1-\mu^*)^2 v_{bb}. \tag{7}$$

The payoff $\bar{v}$ is calculated as the weighted average of the payoffs in all types of pair's status. In other words, $\bar{v}$ is an expected payoff which a player gets in the steady state. We regard $\bar{v}$ as a payoff of game Γ hereafter. Substituting the equation (6) and $\mu^* = 1/2$ into the equation (7), we obtain the payoff as $\bar{v} = 1/2$ in the steady state. We record this fact as the following.

Fact 2 *If all players in the society follow Norm 1 in game* Γ*, the expected payoff which a player gets in the steady state is* $1/2$ *for any* $\delta \in [0, 1)$*.*

Next, we show that Norm 1 is a norm equilibrium of game Γ. As confirmed in subsection 2.4, if all players follow Norm 1, the society is in the steady state $\mu^* = 1/2$. Then, if Norm 1 satisfies requirement (i), it straightforwardly satisfies requirement (ii) by the definition of Norm 1. Thus, we only need to check requirement (i). Let $v_\theta^{'}$ denote the payoff by optimally deviating in each pair's status. Then, the incentive compatible conditions in all types of pair's status are as follows:

$$v_{gg} \geq v_{gg}^{'} = 2(1-\delta) + \delta\{\frac{1}{3}(\frac{1}{2}v_{gg} + \frac{1}{2}v_{gb}) + \frac{2}{3}(\frac{1}{2}v_{bg} + \frac{1}{2}v_{bb})\}, \tag{8}$$

$$v_{gb} \geq v_{gb}^{'} = (1-\delta) + \delta(\frac{1}{2}v_{gg} + \frac{1}{2}v_{gb}), \tag{9}$$

$$v_{bg} \geq v_{bg}^{'} = \delta(\frac{1}{2}v_{bg} + \frac{1}{2}v_{bb}), \tag{10}$$

$$v_{bb} \geq v_{bb}^{'} = -(1-\delta) + \delta\{\frac{1}{3}(\frac{1}{2}v_{gg} + \frac{1}{2}v_{gb}) + \frac{2}{3}(\frac{1}{2}v_{bg} + \frac{1}{2}v_{bb})\}. \tag{11}$$

The inequalities (9) and (11) are satisfied for any $\delta \in [0, 1)$. Substituting (6) into the inequality (8), we obtain $\delta \geq 3/4$. Substituting (6) into the inequality (10), we obtain the same result $\delta \geq 3/4$. Therefore, we have the following fact.

Fact 3 *Norm 1 is a norm equilibrium of game* Γ *if and only if* $\delta \geq 3/4$*.*

Fact 3 shows that Norm 1 is an example of norm equilibria in our model when players are sufficiently patient. Fact 2 also shows that the equilibrium payoff realized under Norm 1 is 1/2. This payoff level is equal to one attainable in our benchmark. Thus, the realized equilibrium payoff is considered to be not too low. Turning our eyes to individual player's behavior, each player follows the same pattern of behavior in the equilibrium. More concretely, pair's status (g, g) is realized in 1/4 of all pairs and the players mutually exert effort, pair's status (b,b) is realized in 1/4 of all pairs and the players mutually shirk, and pair's status (g,b) or (b,g) is realized in 1/2 of all pairs and the players engage in asymmetric punishments. The behavior patterns are stationary in the steady state.

In the next subsections, we present another example of social norms and show that norm equilibria can achieve greater payoffs than 1/2. From now on, the object of our examinations is within the limited class of social norms in which the output function is fixed to be the same as Norm 1. We will mention about the modification of the output function in the Concluding Remarks.

3.3 Norm 2

Norm 1 is a simple but may not be satisfactory from the viewpoint of efficiency. In fact, equilibrium payoff levels can be improved by making some modifications on Norm 1. Norm 2 is such an example of social norms. In Norm 2, the output function is the same as Norm 1 but the transition function is modified as follows.

Norm 2.

$$\tau_i(\theta,y)=\begin{cases}1, & if \quad (\theta,y)=((g,g),\bar{y}),((g,b),\bar{y}),((g,b),\underline{y}),((b,g),\bar{y}),\\ \frac{1}{4}, & if \quad (\theta,y)=((b,b),\underline{y}),\\ 0, & if \quad (\theta,y)=((g,g),\underline{y}),((b,g),\underline{y}),((b,b),\bar{y}).\end{cases}$$

This transition function means the followings. In Norm 1, if both players shirk by following the prescribed actions when the pair's status is (b,b), then the signal $\underline{y}$ arises with probability 1 and both players' statuses remain b. Norm 2 takes relief measures for this phase such that the status b is transferred to g with probability 1/4 even if the signal $\underline{y}$ arises.

When all players follow Norm 2, the transition of status distribution μ_t follows the next equation.

$$\mu_{t+1}=\frac{2}{3}\mu_t^2+\mu_t(1-\mu_t)+\frac{1}{3}\mu_t(1-\mu_t)+\frac{1}{4}(1-\mu_t)^2. \tag{12}$$

We get $\mu^*=3/5$ by substituting the condition $\mu_t=\mu_{t+1}$ into equation (12) and easily confirm that this steady state is stable.

Using $\mu^*=3/5$, we get the following fact.

Fact 4 *If all players in the society follow Norm 2 in game* Γ*, the expected payoff which a player gets in the steady state is* $3/5$ *for any* $\delta \in [0, 1)$*.*

The proof of this fact is relegated to the Appendix A.

We also confirm that Norm 2 is a norm equilibrium of game Γ. We record this fact as the following.

Fact 5 *Norm 2 is a norm equilibrium of game* Γ *if and only if* $\delta \geq 6/7$*.*

The proof of this fact is relegated to the Appendix B.

Facts 4 and 5 show that Norm 2 is also a norm equilibrium and the equilibrium payoff is greater than $1/2$. As the ratio of g players in the steady state increases to $3/5$, the ratio of pairs engaging in mutual cooperation increases while the ratios of the other pairs decrease. The effect raises the equilibrium payoff to $3/5$.

3.4 Norm 3

Equilibrium payoffs can be improved by making other modifications on Norm 1. Norm 3 is an example of such social norms. In Norm 3, the output function is the same as Norm 1 but the transition function is modified as follows.

Norm 3.

$$\tau_i(\theta, y) = \begin{cases} 1, & if \quad (\theta, y) = ((g,g),\bar{y}), ((g,b),\bar{y}), ((g,b),\underline{y}), ((b,g),\bar{y}), \\ \frac{1}{4}, & if \quad (\theta, y) = ((g,g),\underline{y}), \\ 0, & if \quad (\theta, y) = ((b,g),\underline{y}), ((b,b),\bar{y}), ((b,b),\underline{y}). \end{cases}$$

This transition function means the followings. In Norm 1, even if both players exert effort by following the prescribed actions when the pair's status is (g, g), the signal $\underline{y}$ arises with probability $1/3$ and both players' statuses are transferred to b. Norm 3 takes relief measures for this phase such that the status g remains with probability $1/4$.

When all players follow Norm 3, the transition of μ_t follows the next equation.

$$\mu_{t+1} = \frac{3}{4}\mu_t^2 + \mu_t(1 - \mu_t) + \frac{1}{3}\mu_t(1 - \mu_t). \tag{13}$$

We get $\mu^* = 4/7$ by substituting the condition $\mu_t = \mu_{t+1}$ into equation (13) and easily check that this steady state is stable.

Using $\mu^* = 4/7$, we get the following fact.

Fact 6 *If all players in the society follow Norm 3 in game* Γ*, the expected payoff which a player gets in the steady state is* $4/7$ *for any* $\delta \in [0, 1)$.

The proof of this fact is relegated to the Appendix C.

We also confirm that Norm 3 is a norm equilibrium of game Γ. We record this fact as the following.

Fact 7 *Norm 3 is a norm equilibrium of game* Γ *if and only if* $\delta \geq 6/7$.

The proof of this fact is relegated to the Appendix D.

In the same way as Norm 2, the increase of g players raises the equilibrium payoff in Norm 3. From facts 4-7, we find that there exist different channels to improve equilibrium payoffs. We give a detailed consideration of the point in the next section.

4 Discussion

Until now we present three examples of social norms in game Γ, calculate the payoffs attained in norm equilibria, and check the incentive compatible conditions. More generally, what social norms can be sustained in norm equilibria and what range of payoffs can be attained in equilibria? In this section we summarize our answers to these two questions. Now we need to make an excuse for the following statements. The results we state here are not given through mathematical analysis. They are investigated by using a programming of Mathematica.

First, let us consider the case where equilibrium payoffs are greater than or equal to 1/2. We fix the output function as Norm 1, 2, and 3 prescribe and make modifications on the transition function. For convenience of explanation, using three parameters α, β, and γ, we express the transition function as follows:

$$\tau_i(\theta, y) = \begin{cases} 1, & if \quad (\theta, y) = ((g,g),\bar{y}),((g,b),\bar{y}),((g,b),\underline{y}),((b,g),\bar{y}), \\ \alpha, & if \quad (\theta, y) = ((g,g),\underline{y}), \\ \beta, & if \quad (\theta, y) = ((b,g),\underline{y}), \\ \gamma, & if \quad (\theta, y) = ((b,b),\underline{y}), \\ 0, & if \quad (\theta, y) = ((b,b),\bar{y}). \end{cases}$$

As confirmed in the previous section, making adjustments to the transition probabilities in the phases $((g,g),\underline{y})$ and $((b,b),\underline{y})$ improve efficiency. In addition, we confirm that increasing the transition probability in the phase $((b,g),\underline{y})$ leads to improving efficiency. Notice that these probabilities in such phases are denoted by α, β, and γ in the above expression $\tau_i(\theta, y)$. Then, we confirm that the incentive compatible conditions are satisfied under the following inequalities.

$$2(\alpha+\beta)+\frac{3}{2}\gamma < 1, \tag{14}$$

$$\alpha < \frac{1}{2}, \quad \beta < \frac{1}{4}, \quad \gamma < \frac{2}{3}. \tag{15}$$

We also find that the value of the equilibrium payoff approaches 2/3 as the left-hand side of inequality (14), i.e., $2(\alpha+\beta)+\frac{3}{2}\gamma$ approaches 1 under the condition that three inequalities of the condition (15) are satisfied. There exist various types of transition function in which the left-hand side of inequality (14) is near 1. If we modify the transition function to achieve a greater payoff than or equal to 2/3, then the incentive compatible conditions cannot be satisfied. Thus, we confirm that payoffs in the range $[1/2, 2/3)$ can be realized in equilibria.

Next, let us consider the case where equilibrium payoffs are lower than 1/2. As the first case, we fix the output function and only modify the transition function. Using a parameter η, we express the transition function as follows:

$$\tau_i(\theta, y) = \begin{cases} 1, & if \quad (\theta, y) = ((g,g),\bar{y}),((g,b),\bar{y}),((b,g),\bar{y}), \\ \eta, & if \quad (\theta, y) = ((g,b),\underline{y}), \\ 0, & if \quad (\theta, y) = ((g,g),\underline{y}),((b,g),\underline{y}),((b,b),\underline{y}),((b,b),\bar{y}). \end{cases}$$

Then, we confirm that the equilibrium payoff decreases as the probability η decreases from 1 and the equilibrium payoff approaches 0 as η approaches 1/2. Thus, we confirm that payoffs in the range $[0, 1/2]$ can be realized in equilibria. We summarize the above two cases as the following fact.

Fact 8 *Any equilibrium payoff* $\bar{v} \in [0, 2/3)$ *can be realized in norm equilibria in game* Γ.

From the above discussion, we get the following implications about the relationship between equilibrium payoffs and modifications of the transition function. Hypothetically, we set Norm 1 as a reference point. Then, in order to increase equilibrium payoffs, some relief measures need to be taken in the phase where a player's status remains or becomes b in the next period. In other words, some forgiveness is needed to increase equilibrium payoffs. On the other hand, in order to decrease equilibrium payoffs, we need to add some measures which a player's status is transferred to b with a positive probability in the phase where it remains g in the next period. This direction of modifications means imposing more severe punishments. In addition, these types of social norms are similar to the contagious strategies, in which a player always shirks once she encounters a deviator. In the context of our model, the description corresponds to one that if g player encounters b player, then her status is transferred to b with a positive probability. We find it to be none other than what the transition function of the second case means. As stated in the previous section, there exists a correlation between equilibrium payoff levels and the status distribution in our model. The relationship described above is a result of this fact.

5 Concluding Remarks

This paper presents a variant of random matching game. We incorporate a prisoner's dilemma game whose monitoring structure has two public signals into the stage game of the random matching game. In addition, we set up an environment where local information processing is available in the society and players can make use of social norms. Then, players exchange noisy information and decisions are made on noisy information. Through examinations of numerical examples, we find norm equilibria which can sustain partial cooperation even in such environments. In our model, multiple norm equilibria exist and various equilibrium payoffs are realized. We also confirm that the society is in the steady state and all players keep following the same pattern of behavior in an equilibrium.

There exist many remaining issues of this paper. We enumerate them in order. In this paper, the payoff matrix of the stage game or the probability distribution of signals in the stage game are fixed. Not mentioned in the text, we investigate a few cases where the payoff matrixes or the probability distributions of the signals slightly change. Within the scope of our investigation, we confirm that our findings hold. However, we do not get any results to what extent the findings hold. Further investigations are needed to answer this issue. In addition, consideration of our model is confined to be through examinations of numerical examples. We will explore the way to use mathematical analysis in our future research.

We consistently examine a class of social norms in which players' choices are restricted to pure actions. Then, there exists an upper bound to the value of the equilibrium payoff and the maximum attainable payoff is lower than the efficient level. Our main purpose is to find a cooperative equilibrium under an environment where monitoring structure between paired players is imperfect. For the present we target social norms as simple as possible in order to pursue the purpose. Moreover, simplicity of prescribing actions and state transitions is important in the light of the nature of social norms. However, whether the efficient payoff can be sustained in equilibria is also a significant subject. To answer the question, we need to broaden the class of social norms which we investigate. In particular, extending the output function to include mixed or correlated strategies requires consideration.

Our study is based on the premise that all players know the status distribution which conveys information about the prevailing social norms to players. The assumption of our model is crucial because the status distribution, in principle, changes over time. Some mechanisms which make such information transmission possible are needed. Information which players can use is plausible to be decentralized if at all possible. It is reasonable to assume that players have a memory for statuses of the opponents matched in the past. If the number of opponents' statuses in a player's memory are sufficiently large, the distribution of the opponent status levels in her memory is near the status distribution in the society. We may utilize the relationship between the two distributions. It is a major remaining issue in our study to formalize the idea.

Our study also depends on the premise that a player's status cannot be manipulated by the owner. Moreover, a transition of a player's status need to be properly operated. However, we do not present an answer how such local information processing is implemented. This issue also requires consideration.

Appendix

A Proof of Fact 4

The payoffs in all types of pair's status are related to ones in other pair's statuses as follows:

$$v_{gg} = (1-\delta) + \delta\{\frac{2}{3}(\frac{3}{5}v_{gg} + \frac{2}{5}v_{gb}) + \frac{1}{3}(\frac{3}{5}v_{bg} + \frac{2}{5}v_{bb})\}, \tag{16}$$

$$v_{gb} = 2(1-\delta) + \delta(\frac{3}{5}v_{gg} + \frac{2}{5}v_{gb}), \tag{17}$$

$$v_{bg} = -(1-\delta) + \delta\{\frac{1}{3}(\frac{3}{5}v_{gg} + \frac{2}{5}v_{gb}) + \frac{2}{3}(\frac{3}{5}v_{bg} + \frac{2}{5}v_{bb})\}, \tag{18}$$

$$v_{bb} = \delta\{\frac{1}{4}(\frac{3}{5}v_{gg} + \frac{2}{5}v_{gb}) + \frac{3}{4}(\frac{3}{5}v_{bg} + \frac{2}{5}v_{bb})\}. \tag{19}$$

Solving the simultaneous linear equations (16) - (19), the payoffs in all types of pair's status are as follows:

$$\begin{aligned}&(v_{gg}, v_{gb}, v_{bg}, v_{bb})\\ &= (\frac{30-23\delta+2\delta^2}{15(2-\delta)}, \frac{20-16\delta-\delta^2}{5(2-\delta)}, \frac{-30+47\delta-8\delta^2}{15(2-\delta)}, \frac{\delta(-1+4\delta)}{5(2-\delta)}).\end{aligned} \tag{20}$$

Substituting the equation (20) and $\mu^* = 3/5$ into the equation (7), we obtain the payoff as $\bar{v} = 3/5$.

B Proof of Fact 5

As confirmed in the subsection 3.3, if all players follow Norm 2, the society is in the steady state $\mu^* = 3/5$. If requirement (i) is satisfied, requirement (ii) is straightforwardly satisfied by the definition of Norm 2. Thus, we only check requirement (i). The incentive compatible conditions in all types of pair's status are as follows:

$$v_{gg} \geq v'_{gg} \equiv 2(1-\delta) + \delta\{\frac{1}{3}(\frac{3}{5}v_{gg} + \frac{2}{5}v_{gb}) + \frac{2}{3}(\frac{3}{5}v_{bg} + \frac{2}{5}v_{bb})\}, \quad (21)$$

$$v_{gb} \geq v'_{gb} \equiv (1-\delta) + \delta(\frac{3}{5}v_{gg} + \frac{2}{5}v_{gb}), \quad (22)$$

$$v_{bg} \geq v'_{bg} \equiv \delta(\frac{3}{5}v_{bg} + \frac{2}{5}v_{bb}), \quad (23)$$

$$v_{bb} \geq v'_{bb} \equiv -(1-\delta) + \delta\{\frac{1}{3}(\frac{3}{5}v_{gg} + \frac{2}{5}v_{gb}) + \frac{2}{3}(\frac{3}{5}v_{bg} + \frac{2}{5}v_{bb})\}. \quad (24)$$

The inequalities (22) and (24) are satisfied for any $\delta \in [0,1)$. Substituting (20) into the inequalities (21) and (23), we obtain the same result $\delta \geq 6/7$.

C Proof of Fact 6

The payoffs in all types of pair's status are described as follows:

$$v_{gg} = (1-\delta) + \delta\{\frac{3}{4}(\frac{4}{7}v_{gg} + \frac{3}{7}v_{gb}) + \frac{1}{4}(\frac{4}{7}v_{bg} + \frac{3}{7}v_{bb})\}, \quad (25)$$

$$v_{gb} = 2(1-\delta) + \delta(\frac{4}{7}v_{gg} + \frac{3}{7}v_{gb}), \quad (26)$$

$$v_{bg} = -(1-\delta) + \delta\{\frac{1}{3}(\frac{4}{7}v_{gg} + \frac{3}{7}v_{gb}) + \frac{2}{3}(\frac{4}{7}v_{bg} + \frac{3}{7}v_{bb})\}, \quad (27)$$

$$v_{bb} = \delta(\frac{4}{7}v_{bg} + \frac{3}{7}v_{bb}). \quad (28)$$

Solving the simultaneous linear equations (25) - (28), the payoffs in all types of pair's status are as follows:

$$(v_{gg}, v_{gb}, v_{bg}, v_{bb}) = (\frac{-42+31\delta+3\delta^2}{-14(-3+2\delta)}, \frac{42-40\delta+2\delta^2}{-7(-3+2\delta)}, \frac{(-7+3\delta)(-3+4\delta)}{7(-3+2\delta)}, \frac{4\delta(-3+4\delta)}{7(-3+2\delta)}). \quad (29)$$

Substituting the equation (29) and $\mu^* = 4/7$ into the equation (7), we obtain the payoff as $\bar{v} = 4/7$.

D Proof of Fact 7

As confirmed in the subsection 3.4, if all players follow Norm 3, the society is in the steady state $\mu^* = 4/7$. If requirement (i) is satisfied, requirement (ii) is straightforwardly satisfied.

Thus, we only check requirement (i). The incentive compatible conditions in all types of pair's status are as follows:

$$v_{gg} \geq v'_{gg} \equiv 2(1-\delta) + \delta\{\frac{1}{2}(\frac{4}{7}v_{gg} + \frac{3}{7}v_{gb}) + \frac{1}{2}(\frac{4}{7}v_{bg} + \frac{3}{7}v_{bb})\}, \tag{30}$$

$$v_{gb} \geq v'_{gb} \equiv (1-\delta) + \delta(\frac{4}{7}v_{gg} + \frac{3}{7}v_{gb}), \tag{31}$$

$$v_{bg} \geq v'_{bg} \equiv \delta(\frac{4}{7}v_{bg} + \frac{3}{7}v_{bb}), \tag{32}$$

$$v_{bb} \geq v'_{bb} \equiv -(1-\delta) + \delta(\frac{1}{3}(\frac{4}{7}v_{gg} + \frac{3}{7}v_{gb}) + \frac{2}{3}(\frac{4}{7}v_{bg} + \frac{3}{7}v_{bb})). \tag{33}$$

From these inequalities, we obtain the result that all conditions are satisfied if $\delta \geq 6/7$.

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